THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 6 (March 11)

Monotone Convergence Theorem. *A monotone sequence of real number is convergent if and only if it is bounded. Furthermore,*

(a) If (x_n) *is a bounded increasing sequence, then* $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.

(b) If (y_n) *is a bounded decreasing sequence, then* $\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$.

Example 1. Let $Z = (z_n)$ be the sequence of real numbers defined by

$$z_1 \coloneqq 1, \quad z_{n+1} \coloneqq \sqrt{2z_n} \quad \text{for } n \in \mathbb{N}.$$

Show that $\lim(z_n) = 2$.

Example 2 (*Euler number* e)**.** Let $e_n \coloneqq (1 + 1/n)^n$ for $n \in \mathbb{N}$. Show that the sequence $E = (e_n)$ is bounded and increasing, hence convergent. The limit of this sequence is called the *Euler number*, and it is denoted by e.

Definition. Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a **strictly increasing** sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$$

is called a **subsequence** of X.

Theorem. *If a sequence* $X = (x_n)$ *of real numbers converges to a real number* x*, then any subsequence* $X' = (x_{n_k})$ *of* X *also converges to* x.

Example 3. By considering subsequences, deduce the following limits.

(a) $\lim(b^n) = 0$ if $0 < b < 1$.

(b) $\lim(c^{1/n}) = 1$ if $c > 1$.

Classwork

1. Establish the convergence or the divergence of the sequence (x_n), where

$$x_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

Solution. Note that, for all $n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} - x_n &= \left(\frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{(2n+1)(2n+2)} > 0, \end{aligned}$$

and

$$x_n \leq \underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ terms}} = 1.$$

Therefore (x_n) is increasing, bounded above, and hence convergent by Monotone Convergence Theorem. ◀

2. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. (Hint: $1 + 2\sqrt{p}$ is one upper bound.)

Solution. Note $y_2 = \sqrt{p + \sqrt{p}} > \sqrt{p} = y_1$. Suppose $y_{k+1} > y_k$ for some $k \in \mathbb{N}$. Then

$$y_{k+2} = \sqrt{p + y_{k+1}} > \sqrt{p + y_k} = y_{k+1}.$$

By induction, $y_{n+1} > y_n$ for all $n \in \mathbb{N}$.

Note $y_1 = \sqrt{p} < 1 + 2\sqrt{p}$. Suppose $y_k < 1 + 2\sqrt{p}$ for some $k \in \mathbb{N}$. Then

$$y_{k+1} = \sqrt{p + y_k} < \sqrt{p + 1 + 2\sqrt{p}} = \sqrt{(1 + \sqrt{p})^2} < 1 + 2\sqrt{p}.$$

By induction, $y_n < 1 + 2\sqrt{p}$. for all $n \in \mathbb{N}$.

The sequence (y_n) is thus increasing and bounded above. By Monotone Convergence Theorem, $y := \lim(y_n)$ exists. Since $y_{n+1} = \sqrt{p + y_n}$, we have

$$y = \sqrt{p + y} \implies y^2 - y + p = 0 \implies y = \frac{1}{2}\left(1 \pm \sqrt{1 + 4p}\right).$$

Since $y_n > 0$ for all $n \in \mathbb{N}$, we have $y \geq 0$ and hence $y = \frac{1}{2}\left(1 + \sqrt{1 + 4p}\right)$. ◀